J. Fluid Mech. (1981), *vol.* 103, *pp.* 87–115
Printed in Great Britain

On the theory of nonlinear wave–wave interactions among geophysical waves

By P. RIPA†

The Joint Institute for the Study of the Atmosphere and the Ocean, PMEL and UW, 3711 15th Ave NE, Seattle, Wa 98105

(Received 8 May 1978 and in revised form 27 May 1980)

The problem of nonlinear wave–wave interactions is reformulated, in a Eulerian framework, for two classical geophysical systems: barotropic Rossby waves and internal gravity waves on a vertical plane. The departure of the dynamical fields from the equilibrium state is expanded in the linear-problem eigenfunctions, using their properties of orthogonality and completeness. The system is then completely described by the expansion amplitudes, whose evolution is controlled by a system of equations (with quadratic nonlinearity) which is an *exact* representation of the original model equations. There is no *a priori* need for the usual multiple-time-scale analysis, or any other perturbation expansion, to develop the theory. These or other approximations (like truncation of the expansion basis or the Boltzmann equation for a stochastic description) can, if desired, be performed afterwards.

The evolution of the system is constrained mainly by the conservation of energy E and pseudo-momentum P, properties related to time and space homogeneity of the model equations. Conservation of E and P has, in turn, some interesting consequences: (*a*) a generalization of Fjortoft's theorem, (*b*) a class of exact nonlinear solutions (which includes the system of one single wave), and (*c*) conservation of E and P in an arbitrarily truncated system (which is useful in the development of approximations of the problem).

The properties of all possible resonant triads are shown and used to estimate the order of magnitude of off-resonant coupling coefficients.

The results are used in two different problems: the stability of a single wave (maximum growth rates are evaluated in both the strong- and weak-interactions limits) and the three-wave system. The general solution (for any initial condition and for both the resonant and off-resonant cases) of the latter is presented.

1. Introduction

There is a growing interest in understanding the mechanism of nonlinear wave–wave interactions in geophysical fluids. This is evident in the number of recent papers devoted to this problem, in both deterministic (e.g. Orlanski & Cerasoli 1980) and stochastic (e.g. Holloway & Hendershott 1977; McComas & Bretherton 1977) systems.

Most studies of nonlinear wave–wave interactions are based on a perturbation expansion in a 'small' parameter, which is taken to be some sort of overall normalization of the wave amplitudes. This expansion, in time, requires a multiple time-scale

† Present address: CICESE, ap. postal 2732, Ensenada, Baja California, Mexico.

0022–1120/81/4635–3170 $02.00

BRW	IGW
Notation	
$\mathbf{v} = (u, v) = (-\partial_y, \partial_x)\,\psi$	
$D_t = \partial_t + \mathbf{v}\cdot\nabla$	
$\xi = (\Delta - K_R^2)\,\psi$	$\rho = 1 + g^{-1}N^2(\eta - y)$
Model equations	
$D_t(\xi + f) = 0$	$D_t\Delta\psi + N^2\partial_x\eta = 0, D_t\rho = 0$
Integrals of motion	
$E = \frac{1}{2}\langle(\nabla\psi)^2 + K_R^2\psi^2\rangle$	$\frac{1}{2}\langle(\nabla\psi)^2 + N^2\eta^2\rangle$
	Energy
$U = \langle u - \beta^{-1}f K_R^2\psi\rangle$	$\langle u\rangle$
	Momentum
$A[F] = \langle F(\xi + f) - F(f)\rangle$	$\langle F(\rho)\,\Delta\psi\rangle$
	(Any F)
$P = -\frac{1}{2}\beta^{-1}\langle\xi^2\rangle$	$\langle\eta\Delta\psi\rangle$
	Pseudo-momentum

TABLE 1

analysis, due to the existence of resonant interactions. The algebra to develop the theory in this way is in most cases quite cumbersome and, as a consequence, the expansion is usually performed up to the first nonlinear order. Moreover, this expansion is clearly not very useful if one were to use the theory with finite amplitude 'waves' (for instance, to study the transition between wave and turbulent regimes).

It will be shown here that, for many geophysical systems, it is not necessary to perform this perturbation expansion. The nonlinear evolution equation can be easily obtained from the Eulerian model equations and is an *exact* representation of the latter. The method is based on the orthogonality and completeness of the linear-problem eigensolutions, which follow from the hermiticity of the model equations. The conservation of energy and pseudo-momenta and its consequences are explicit in this formalism. Finally, any approximation (like the multiple time-scale/perturbation expansion or the Boltzmann stochastic equation) can, if desired, be performed *a posteriori*.

The author has applied this study to several geophysical systems. In this paper are presented the results for the problems of barotropic Rossby waves in the mid-latitude beta-plane (BRW) and internal gravity waves on a vertical plane (IGW). The properties of both systems are studied 'in parallel', in order to stress their formal similarity. Other cases studied are equatorial waves (Ripa 1980*a*) and 3-D inertia–gravity/planetary waves on the beta-plane and on the sphere (Ripa 1980*b*).

Section 2 is devoted to the description of the Eulerian model equations and conservation laws. The notation and main results are summarized in table 1. The wave expansion and the corresponding evolution equation are developed in § 3. The kinematic (wavenumbers and frequencies) and dynamic (interaction coefficient) properties of all possible resonant triads are shown in § 4 and used to estimate the magnitude of the coupling coefficients of the off-resonant triads.

The results of these three sections are applied, in § 5, to two simple nonlinear

problems: the stability of a single wave (both in the strong- and weak-interactions limits) and the study of the three-wave system. The latter problem may be taken as a first step in the understanding of nonlinear wave–wave interactions (Bretherton 1964). Furthermore, the same set of equations appears in other nonlinear systems (for example, capillary–gravity waves in a three-layer fluid, Craik & Adam 1979). McGoldrick (1965) found a particular solution of the *resonant* three-wave problem, that is often quoted in the literature. The general solution (for any initial condition and valid for both resonant and off-resonant cases) is presented here, along with a description of its characteristics. The solution is also valid (in a different range of the parameters) for 'explosive' systems like that of Craik & Adam (1979). Details of the algebra are given in the appendix. Finally, § 6 is devoted to general discussion.

2. Model equations and conservation laws

Two different geophysical systems, Bartotropic Rossby Waves (BRW) and Internal Gravity Waves (IGW), are considered in this paper. Their properties are discussed 'in parallel' in order to stress their formal similarity, in spite of the fact that they are quite different physical systems. The notation, model equations and conservation laws are summarized in table 1.

For BRW, i.e. quasi-geostrophic flow in a one-layer model, (Longuet-Higgins & Gill 1967), x and y are the zonal and meridional position co-ordinates, $f(=f_0+\beta y)$ is the Coriolis parameter and $K_R(=f_0/\sqrt{(g'H)})$ is the inverse of the deformation radius where H is the mean depth of the layer and g' is effective gravity. The model equation expresses the conservation of the potential vorticity $(\xi+f)$ following a fluid column. This layer can be thought to be between two deep layers with no horizontal motion. Depending on the density ratio of the three layers, this system may be taken as describing quasi-geostrophic flow in either the external ($g' = g$) or one of the internal ($g' \ll g$) vertical modes, under the approximation of neglecting the interactions of this mode with the others (Ripa 1980*b*).

For IGW on a vertical plane (Orlanski 1973), x and y are the horizontal and vertical position co-ordinates, N is the Brunt–Väisälä frequency (taken as constant) and ρ is the total density divided by the mean (over the entire plane) density. The model equations express the conservation of density and the changes of vorticity due to the sloping of the isopycnals.

Particular attention is paid to the effect of nonlinear terms in the free evolution of the system, thereby excluding any external forcing. Likewise, boundary effects are not considered, formally taking the space domain as unbounded and requiring the dynamical fields to be bounded at infinity.

The evolution of the system is constrained by the existence of several conservation laws (see table 1). First, each fluid parcel conserves its potential vorticity (density), in the case of BRW (IGW). Using incompressibility, this law can be shown to be equivalent to the existence of infinite integrals of motion, namely the functionals $A[F]$, where F is *any* function of the potential vorticity (density). In addition to the $A[F]$, there are two more integrals of motion: the energy E and the momentum U. The angle brackets in the equations of table 1 are proportional to the integration over the entire space, for example, for some initial conditions they may be defined as the integral itself,

$$\langle \ldots \rangle = \int d\mathbf{x} \ldots, \tag{2.1}$$

whereas in some other cases it is better to define them as the *average*

$$\langle \ldots \rangle = \lim \left[\int d\mathbf{x} \ldots / \int d\mathbf{x} \right], \tag{2.2}$$

with the limit taken as the domain of both integrations tends to infinity.

The energy E is a *quadratic* functional of the dynamical fields and the consequences of its conservation are quite straightforward in an expansion formalism such as the one that is developed in the following section. The momentum U and the A's are, though, *linear* (to lowest order) functionals of the dynamical fields, and the consequences of their conservation are more obscure, especially in a framework such as that of § 3. However, there is one (and only one) combination of U and the A's which is *quadratic*, namely the *pseudo-momentum* along the x-direction,

$$P = U + A[F_0], \tag{2.3}$$

where $F_0 = -(\xi+f)^2/2\beta (F_0 = g\rho/N^2)$ for BRW (IGW).

The pseudo-momentum P should not be confused with the momentum U, as was clearly pointed out by Andrews & McIntyre (1978). For instance, under the Galilean transformation $x \to x + U't$, it is (using the definition (2.2)) $E \to E + UU' + \text{const.}$, $U \to U + U'$ but $P \to P$, as can be easily proved from the expressions in table 1.

E and P are the only integrals of motion of the fully nonlinear systems which are quadratic in the departure from the equilibrium state. For BRW, P is negative definite, whereas for IGW, P may have either sign. It will be shown in § 3 that conservation of E and P has several interesting consequences on the nonlinear evolution of the system.

Milder (1976) proposed a different quantity (quadratic and positive definite) as a measure of IGW saturation, noting that it is an integral of motion in the *linear* problem. However, it is easy to show (see § 5) that nonlinear interactions (even the resonant ones) violate the conservation of this quantity, and therefore its physical meaning is not very clear.

The expression for the pseudo-momentum in terms of the Eulerian variables is quite different for both systems: P is proportional to the mean of the square of the perturbation of potential vorticity (usually called enstrophy) for BRW, and to the correlation between vorticity and density perturbation for IGW. However, in the expansion formalism developed in the following section, P has the same expression in both systems. This does not happen, obviously, by chance. It can be shown that P is related to the invariance of both systems under translations along x.

If the evolution equations of a certain continuous system can be derived from a variational principle, then the integrals of motion and symmetries of the problem can be related using Noether's theorem (see Gelfand & Fomin 1963, p. 177). For the problems of this paper, this can be done in the *Lagrangian* description of the system, that is, using as dependent variables the position of the fluid particles

$$\mathbf{x} = \mathbf{x}(\mathbf{x}', t),$$

where the label $\mathbf{x}'$ is taken to be the *equilibrium* position. The details of the derivation of the conservation laws from the symmetries of the systems will be presented elsewhere (Ripa 1980*b*). Only the final results are presented below.

(*a*) Time homogeneity (i.e. the fact that a solution of the problem translated in t is also a solution) is related to conservation of energy.

ϕ	BRW ψ	IGW $\begin{pmatrix}\psi\\ \eta\end{pmatrix}$	Dynamical variables
	$\mathbf{k} = K(\cos\theta, \sin\theta)$ Wavenumber $\mathbf{s} = \mathbf{k}/\omega = (s, s\tan\theta)$ Slowness		
$s(K)$	$-\beta^{-1}I^2\,(I^2 = K^2 + K_R^2)$	K/N	x-slowness
$\omega(K,\theta)$	$-\beta K\cos\theta/I^2$	$N\cos\theta$	Eigenfrequency
$\psi(K,\theta)$	$I^{-1}\exp(i\mathbf{k}\,.\,\mathbf{x})$	$(\sqrt{2}\,K)^{-1}\begin{pmatrix}-1\\ s\end{pmatrix}\exp(i\mathbf{k}\,.\,\mathbf{x})$	Eigenfunction
σ_i^{jk}	$\hat{\mathbf{z}}\,.\,\mathbf{k}_j\times\mathbf{k}_k(I_iI_jI_k)^{-1}$ $\times(K_j^2-K_k^2)$	$\sqrt{2}\sin\frac{1}{2}(\theta_i-\theta_j)\sin\frac{1}{2}(\theta_j-\theta_k)$ $\times\sin\frac{1}{2}(\theta_k-\theta_i)\,(K_j-K_k)$	Coupling coefficients
Γ	$[\Sigma\,(s_i^2-s_js_k)\,(1-\beta^{-1}K_R^2/s_i)]^{\frac{1}{2}}$	$-8^{-\frac{1}{2}}\Sigma\sin\theta_i\,\Sigma s_i$	Interaction coefficients

TABLE 2

(*b*) The system is invariant under a general (volume preserving) change of the label $\mathbf{x}'$. This results in the conservation of potential vorticity (density) for BRW (IGW).

(*c*) The conserved particle property mentioned above can be used to define one of the equilibrium co-ordinates, say y', for all fluid elements. Incompressibility can then be used to calculate the other equilibrium co-ordinate, x', up to an arbitrary function of y'. This symmetry yields the conservation of the functionals $A[F]$. The particular one of (2.3) (i.e. $A[F_0] = P - U$) corresponds to a rigid translation of the equilibrium co-ordinate x' for all fluid particles.

(*d*) Finally, the system is invariant under a rigid translation along the position co-ordinate x. This symmetry yields conservation of $2U - P$.

Note that x-homogeneity is responsible for the existence of the integrals of motion in both (*c*) and (*d*), for example, for the conservation of both the momentum and the pseudo-momentum. Conservation of U or P alone is related to simultaneous translations in x and x'.

A pseudo-momentum (quadratic, to lowest order) is then expected to be conserved along each homogeneous co-ordinate of any geophysical fluid system. For the systems discussed in this paper, the y-co-ordinate is *not* homogeneous (this can be seen in the expressions of the conservation laws), even though it does not appear explicitly in the model equations used.

3. The wave expansion

Let $\phi(\mathbf{x}, t)$ be a (vector) field that groups the dynamical variables, defined in such a way that the reference state is given by $\phi = 0$ (see table 2). The model equations (table 1) can be written in the form

$$(\partial_t + \mathbf{v}\,.\,\nabla)\,D\phi - L\phi = 0, \tag{3.1}$$

where

$$D = K_R^2 - \Delta \quad \text{(BRW)},$$

$$D = \begin{pmatrix}-\Delta & 0\\ 0 & N^2\end{pmatrix} \quad \text{(IGW)},$$

and

$$L = \beta \partial_x \quad \text{(BRW)},$$

$$L = N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \quad \text{(IGW)}.$$

The operator D is hermitian and L is skew-hermitian, i.e.

$$\langle V_1^\dagger D V_2 \rangle = \langle V_2^\dagger D V_1 \rangle^*, \quad \langle V_1^\dagger L V_2 \rangle = -\langle V_2^\dagger L V_1 \rangle^*, \tag{3.2}$$

where $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$ are any two (complex) vector fields in the same space as ϕ, $\dagger$ denotes transpose and complex conjugate and * means complex conjugate. In addition, D is positive definite. These properties imply that the eigenvalues ω_a of the equation

$$(L + i\omega_a D)\,\phi_a(\mathbf{x}) = 0 \tag{3.3}$$

are real and that the eigenvectors $\phi_a(\mathbf{x})$ corresponding to different eigenvalues are D-orthogonal. In fact, all eigenvectors can be D-orthonormalized, i.e.

$$\langle \phi_a^\dagger(\mathbf{x})\, D \phi_b(\mathbf{x}) \rangle = \delta(a, b), \tag{3.4}$$

where $\delta(a, b) = 0$ if $a \neq b$. Note that

$$E = \tfrac{1}{2}\langle \phi^\dagger D \phi \rangle, \tag{3.5}$$

i.e. the energy integral defines the metric in the space spanned by the $\{\phi_a\}$.

The basis $\{\phi_a\}$ is also *complete*, namely

$$\sum_a \phi_a(\mathbf{x})\, D \phi_a^\dagger(\mathbf{u}) = I \delta(\mathbf{x} - \mathbf{u}), \tag{3.6}$$

where I is the unit matrix in the space of the components of ϕ and

$$\sum_a f(a)\, \delta(a, b) = f(b).$$

This basis can then be used to expand the dynamical fields in the form

$$\phi(\mathbf{x}, t) = \sum_a Z_a(t)\, \phi_a(\mathbf{x}), \tag{3.7}$$

where, using (3.4), it is

$$Z_a(t) = \langle \phi_a^\dagger(\mathbf{x})\, D \phi(\mathbf{x}, t) \rangle. \tag{3.8}$$

This way, the state of the system is described in terms of the expansion amplitudes $Z_a(t)$ instead of the dynamical fields $\phi(\mathbf{x}, t)$. The evolution equation is obtained by replacing (3.7) in (3.1), and using (3.3) and (3.4), which results in

$$\dot{Z}_a + i\omega_a Z_a = \frac{1}{2} \sum_{bc} \sigma_a^{bc} Z_b^* Z_c^*, \tag{3.9}$$

where the dot means time derivative, and

$$\sigma_a^{bc} = -\langle \phi_a^\dagger (\mathbf{v}_b . \nabla D \phi_c + \mathbf{v}_c . \nabla D \phi_b)^* \rangle. \tag{3.10}$$

Equation (3.9) is the fundamental one of this paper. The eigenvalues† ω_a characterize the linear evolution of the system, and the coupling coefficients σ_a^{bc} provide for the

† The ω_a are frequencies strictly only in the linear problem. For the nonlinear case, the ω_a should be taken as constant coefficients.

nonlinear effects. The physical meaning of σ_a^{bc} is the projection into ϕ_a of the advection of ϕ_c by the velocity field of ϕ_b, plus the advection of ϕ_b by the velocity field ϕ_c.

Two points should be clarified about the summations in (3.7) and (3.9).

(1) Since D and L are real, if (ϕ_a, ω_a) is a solution of (3.3), then $(\phi_a^*, -\omega_a)$ is also a solution. The latter solution is called the 'conjugate' of the former, and labelled by a^*. Solutions a and a^* are *mathematically* independent, but they represent the same *physical* state, because the reality of ϕ requires $Z_{a^*}(t) = Z_a(t)^*$; a condition which is obviously satisfied by the definition (3.8) and preserved by (3.9).

(2) Since D and L are independent of $\mathbf{x}$, one might look for solutions of (3.3) with an $\exp(i\mathbf{k}.\mathbf{x})$ spatial dependence. The label 'a' may be written in the form $a = (\mathbf{k}, m)$, where m is a discrete index that runs over the solutions of (3.3) for each $\mathbf{k}$ (one for BRW and two for IGW). Consequently, it is

$$\sum_a = \frac{1}{(2\pi)^2}\int d\mathbf{k} \sum_m,$$

$$\delta(a, a') = (2\pi)^2 \delta(\mathbf{k}-\mathbf{k}')\,\delta_{mm'},$$

and the $\langle \ldots \rangle$ have the meaning defined in (2.1).

However, for many problems (such as the ones discussed in § 5) the field ϕ can be initially (and, consequently, at all times) expanded in terms of a denumerable set of components. For these problems, Σ_a is a discrete summation, $\delta(a, a')$ is Kronecker's delta, and $\langle \ldots \rangle$ is taken in the sense of (2.2). The Z's and the σ's have then dimensions of velocity and wavenumber. Any coupling coefficient, say σ_a^{bc}, vanishes unless

$$\mathbf{k}_a + \mathbf{k}_b + \mathbf{k}_c = 0. \tag{3.11}$$

The coupling coefficients for the continuous case, on the other hand, are equal to those of the discrete case times $(2\pi)^2 \delta(\mathbf{k}_a + \mathbf{k}_b + \mathbf{k}_c)$.

The expressions for the eigensolutions and coupling coefficients (normalized in the sense of (2.2)) for BRW and IGW are shown in table 2. The coupling coefficients are clearly symmetric under a permutation of the two upper indices. Furthermore, the ϕ_a have been normalized so that the σ's are real. Therefore, there are *three* real coupling coefficients associated to each triad of interacting (i.e., that satisfy (3.11)) components.

Instead of $(\mathbf{k}, m)$, $a = (K, \theta)$ is used here, where θ varies between 0 and 2π, and K (defined so that $|K| = |\mathbf{k}|$) is always positive for BRW but it has the sign of the horizontal phase speed for IGW (the two signs accounting for the two values of m). For instance, the 'conjugate' state (with $-\mathbf{k}$ and $-\omega$) corresponds to the label $a^* = (K, \theta + \pi)$.

Note that the basis $\{\phi_a\}$ is indeed *complete* because there are as many eigensolutions for each $\mathbf{k}$ as ϕ has components. Therefore, there are no extra terms on the right-hand side of (3.7) and (3.9); the latter being an *exact* representation of (3.1). In fact, it is necessary that the eigensolutions of (3.3), including the zero-frequency ones, expand a complete basis even for the *linear* problem to be fully solvable in a wave-expansion formalism.

Consequently, there is no need of a perturbation expansion (nor any assumption on the magnitude of the Z_a) or transformation of the set (3.1) to a higher-order (in ∂_t) equation, in order to obtain the nonlinear wave equations (3.9). Any approximation to the problem (e.g. truncation of the expansion basis) can be done afterwards. In

particular, if a stochastic description of the system is adopted, the Boltzmann equation (Hasselmann 1967; Holloway & Hendershott 1977) can be deduced from (3.9) without any need of a Hamiltonian formulation.

Furthermore, the expression (3.10) for the evaluation of the coupling coefficients is very simple. The particular form of the quadratic term in (3.1) is not important in obtaining (3.10); the only crucial property is the orthogonality condition (3.4) that follows from the hermitian properties of D and L – (3.2).

Although the Lagrangian framework is useful to relate symmetries and integrals of motion, it is inadequate to obtain a nonlinear wave equation such as (3.9) for the following reasons. (*a*) The Lagrangian model equations are second order in time and not in self-adjoint form (and thus there is no equivalent to (3.2)); (*b*) Nonlinearities are of higher order.

The basis $\{\phi_a\}$ is not only complete, but it also yields an *additive* representation of both E and P, namely

$$E = \sum_a{}' E_a, \quad P = \sum_a{}' s_a E_a, \tag{3.12}$$

where $E_a = |Z_a|^2$, s is the x-component of the *slowness* vector† (Hayes 1974, p. 12)

$$\mathbf{s} = \mathbf{k}/\omega, \tag{3.13}$$

and the prime indicates summation over physically different states, i.e. either a or a^* for each conjugate pair (a, a^*). If the expansion (3.7) is made with a different (complete) basis, not only will the frequency term in (3.9) no longer be diagonal, but there would also be double summations in (3.12). In this sense, the $\{\phi_a\}$ might be considered the natural basis to expand the system.

The rate of change of the energy of a particular component is given (taking twice the real part of (3.9) multiplied by Z_a^*) by the equation

$$\dot{E}_a = \sum_{bc} \sigma_a^{bc}\, Re\,(Z_a Z_b Z_c), \tag{3.14}$$

from which it is easy to show that necessary and sufficient conditions for the conservation of E and P are that the coupling coefficients related to any interacting triad, say $(abc) = (123)$, satisfy

$$\begin{aligned} \sigma_1^{23} + \sigma_2^{31} + \sigma_3^{12} &= 0, \\ s_1\sigma_1^{23} + s_2\sigma_2^{31} + s_3\phi_3^{12} &= 0. \end{aligned} \tag{3.15}$$

This is clearly true for those of table 2 because they are of the form $\sigma_a^{bc} \doteq s_b - s_c$ $(abc = 123 + cp)$.

Some interesting consequences can be derived, in this formalism, from the laws of conservation of E and P.

(1) The ratio P/E is a weighted average of the $\{s_a\}$, with weights proportional to the individual energies. Therefore, any flux of energy toward, say, larger s_a, must be balanced by an energy flux toward lower s_a. This is a generalization of Fjortoft's well-known theorem for BRW (for which the statement obviously holds replacing 's_a' by '$|\mathbf{k}_a|$', as can be seen in table 2).

(2) Equations (3.15) imply that if $s_b = s_c$ then $\sigma_a^{bc} = 0$. Therefore, if the initial

† In a multi-dimensional space it is better to work with $\mathbf{s}$ rather than with the phase speed. For instance, the Doppler-shifted frequency for an observer with velocity $\mathbf{U}$ is $\omega(1 - \mathbf{s}\,.\,\mathbf{U})$ and the phase speed in the direction of a unit vector $\hat{\mathbf{e}}$ is $1/\mathbf{s}\,.\,\hat{\mathbf{e}}$ and not $\hat{\mathbf{e}}\,.\,\mathbf{k}\omega/k^2$.

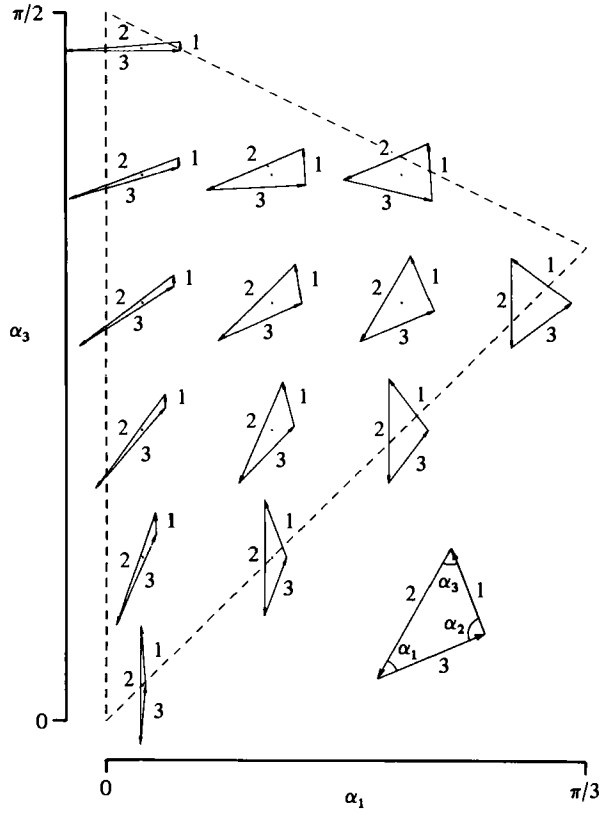

FIGURE 1. Resonant triads of non-divergent Barotropic Rossby waves, in the (α_1, α_3) plane (see definition triangle on figure). The region shown corresponds to $K_1 \leqslant K_3 \leqslant K_2$. The eastward direction (for the vectors) is to the right. All possible resonant triads are obtained by changing the scale of the vectors or changing the sign of the three northward components.

expansion of ϕ involves only components with the same value of s, then the right-hand side of (3.9) vanishes identically (at all times). Consequently, any combination of linear waves with the same phase speed along x and arbitrary amplitudes (in particular, *one single wave*) is an exact solution of the full nonlinear problem. More explicitly, a streamfunction of the form $\psi = F(t-sx, y)$ (BRW and IGW) and isopycnal elevation $\eta = -s\psi$ (IGW) is an exact nonlinear solution provided that $\Delta\psi = (\beta s + K_R^2)\psi$ (BRW) or $\Delta\psi = -s^2N^2\psi$ (IGW).

(3) Equations (3.15) also imply that E and P are also conserved in any 'approximate' system obtained by an *arbitrary truncation* of the expansion basis in (3.7), as long as all corresponding terms in (3.9) are kept.

Finally, (3.15) can be used to determine the relative value of the coupling coefficients of any interacting triad; their order of magnitude is discussed at the end of the following section. The first of equations (3.15) implies that, for a given triad, one of the three σ's has a sign opposite to that of the other two, and consequently is the largest in absolute value. In view of (3.14), it is important to single out this component (for each triad) in order to understand the exchanges of energy among the different components. The second of equations (3.15) implies that the component with largest value of $|\sigma|$,

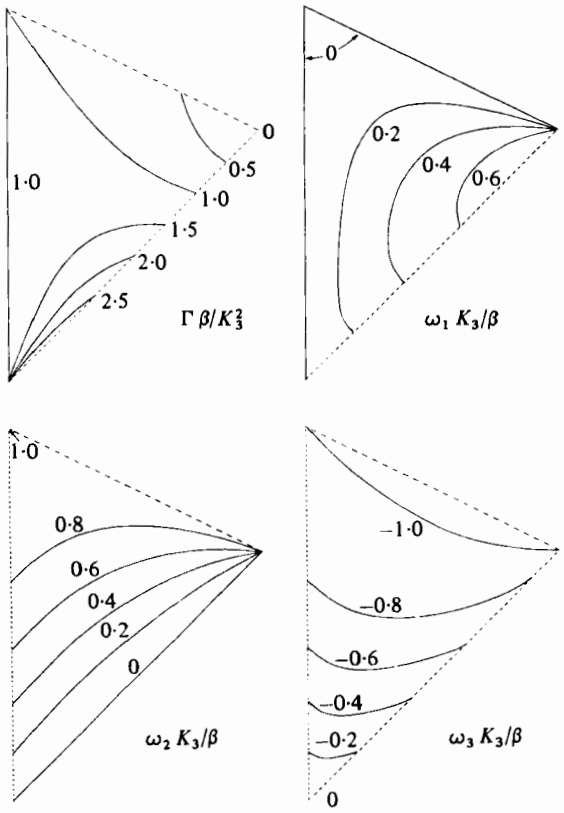

FIGURE 2. Interaction coefficient Γ and frequencies for the triads of figure 1. These variables have been non-dimensionalized using β and the wavenumber of the component with largest $|\omega|$, i.e., K_3. The axes are those of figure 1.

say, wave 3, is the one with *intermediate value of x-slowness* (which is not necessarily equivalent to intermediate value of phase speed along x), i.e.

$$|\sigma_3^{12}| > |\sigma_1^{23}|, \quad |\sigma_2^{31}| \Leftrightarrow (s_2 - s_3)(s_3 - s_1) > 0. \tag{3.16}$$

For instance

$$[\sigma_1^{23}, \sigma_2^{31}]/\sigma_3^{12} = \left\{ \begin{array}{ll} [-1, 0] & \text{if} \quad s_3 = s_1, \\ [-\tfrac{1}{2}, -\tfrac{1}{2}] & \text{if} \quad s_3 = \tfrac{1}{2}(s_1 + s_2), \\ [0, -1] & \text{if} \quad s_3 = s_2. \end{array} \right\} \tag{3.17}$$

4. Resonant triads

If E is 'small', the evolution of the system is controlled, to a first approximation, by the linear part of (3.9); i.e. $Z_a(t) \sim Z_a(0) \exp(-i\omega_a t)$. But then any two components, say, 1 and 2, will modify the evolution of other components (two for BRW and four for IGW) acting as a forcing in the right-hand side of (3.9). For instance, a component with $\mathbf{k}_3 = -\mathbf{k}_1 - \mathbf{k}_2$ 'feels' a harmonic forcing with amplitude $\sigma_3^{12} Z_1^*(0) Z_2^*(0)$ and frequency $-\omega_1 - \omega_2$. This action is expected to be more important if the forcing

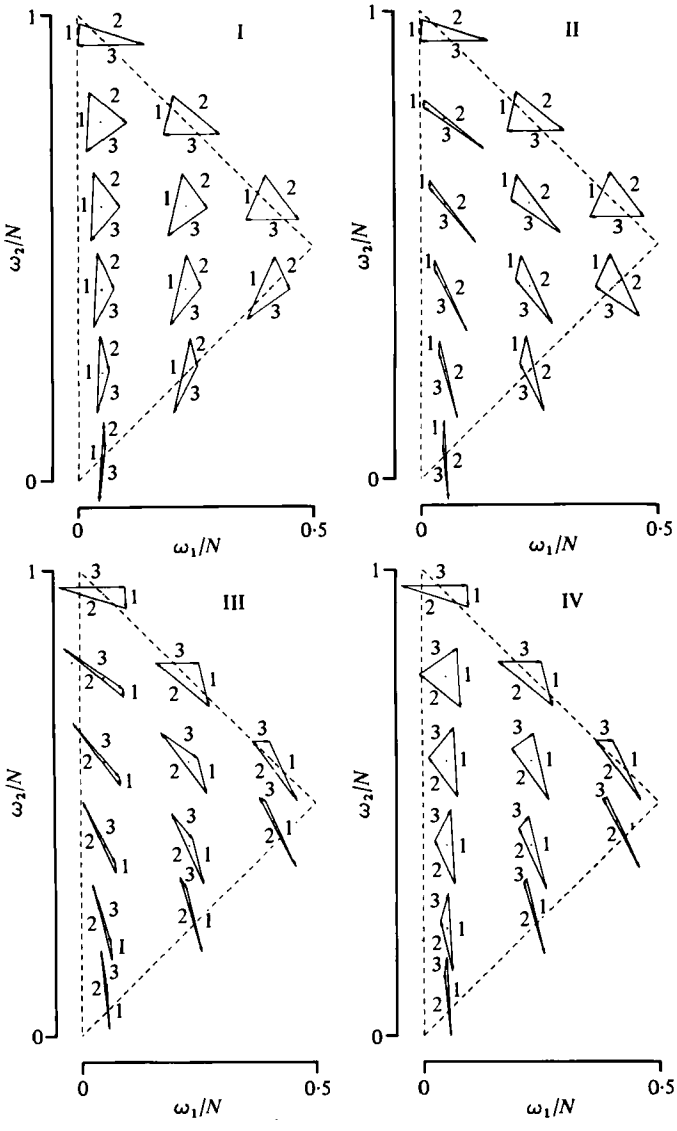

FIGURE 3. Resonant triads of Internal Gravity Waves on a vertical plane in the (ω_1, ω_2) space ($\omega_3 = -\omega_1 - \omega_2$). The four classes differ in the sign of one of the vertical wavenumbers. Four additional classes are obtained changing the sign of the three vertical wavenumbers. All possible triads are obtained by multiplying the three **k** by any real number. The region shown corresponds to the ordering $\omega_1 \leqslant \omega_2 \leqslant -\omega_3$.

frequency coincides with ω_3, i.e., if the three waves satisfy the resonance condition (Phillips 1960)

$$\omega_1 + \omega_2 + \omega_3 = 0. \tag{4.1}$$

A more elegant introduction of the concept of resonant triads is usually made with a multiple time-scale analysis (see, for instance, Bretherton 1964). This analysis can certainly be made starting from the evolution equation (3.9), since it is valid for

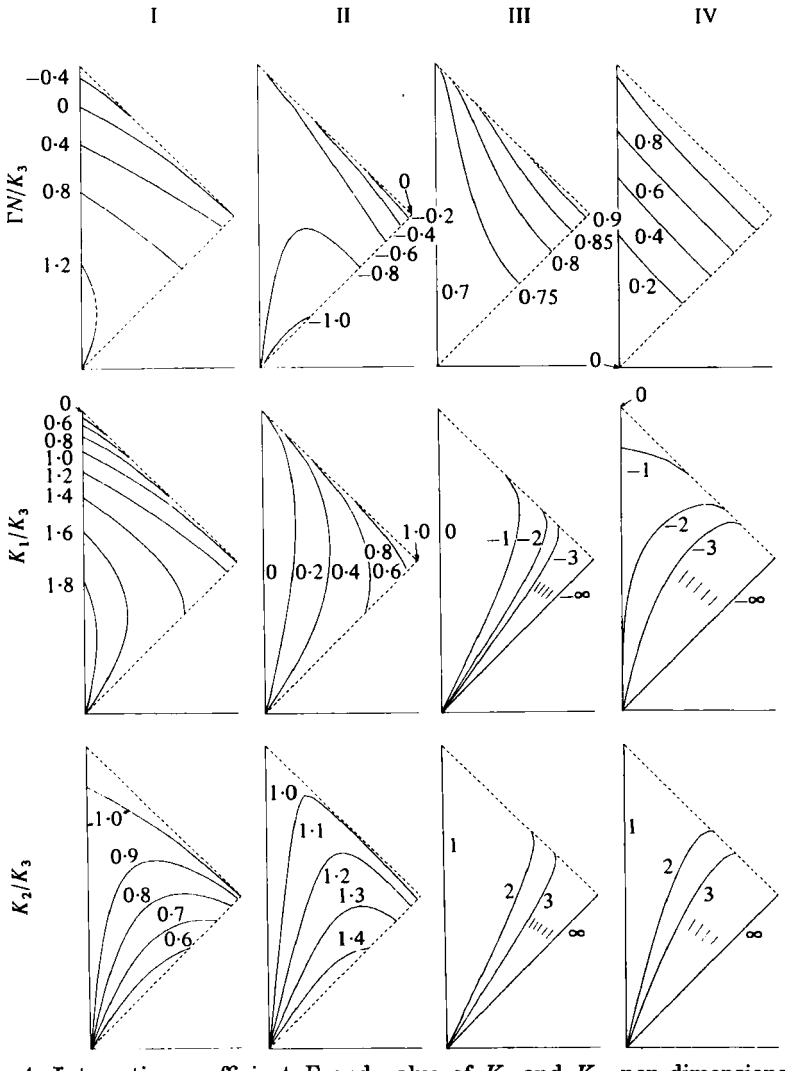

FIGURE 4. Interaction coefficient Γ and value of K_1 and K_2, non-dimensionalized using the Brunt–Väisälä frequency N and the value of K_3 (each K has the magnitude of $|\mathbf{k}|$ and the sign of the rightward phase-speed). The axes are those of figure 3.

motion of any amplitude. A discussion on the limitations of the weak-interactions approximation (which considers only resonant triads) can be found in Holloway & Hendershot (1977), Orlanski & Cerasoli (1980) and Holloway (1980).

Conservation of E and P yield Hasselman's (1967) factorization of the coupling coefficients of a *resonant* triad, namely, if equations (3.11) for $(abc) = (123)$ and (4.1) are valid, then

$$\sigma_1^{23}/\omega_1 = \sigma_2^{31}/\omega_2 = \sigma_3^{12}/\omega_3 = \Gamma_{123}, \tag{4.2}$$

which follows from the fact that (3.15) is still satisfied replacing the σ_a by the ω_a. Equations (3.16) and (4.2) then show that for a resonant triad the component with intermediate value of s, has maximum value of $|\omega|$. The expression for the evaluation

of the interaction coefficients Γ (which have dimensions of slowness) is given in table 2.

The properties of all resonant triads are shown in figures 1 and 2 for non-divergent ($K_R = 0$) BRW and in figures 3 and 4 for IGW. In both cases, component 3 is chosen to be the one with intermediate x-slowness (or, due to the resonance condition, the one with the largest absolute frequency). Thus, without any loss of generality, the figures expand the region

$$\left.\begin{aligned} K_1 \leqslant K_3 \leqslant K_2 \quad &\text{(BRW)}, \\ \omega_1 \leqslant \omega_2 \leqslant -\omega_3 \quad &\text{(IGW)}. \end{aligned}\right\} \tag{4.3}$$

The results are presented in the (α_1, α_3) plane for BRW (where α_a is the angle opposite to $\mathbf{k}_a$ in the triangle formed by the three $\mathbf{k}$'s), and in the (ω_1, ω_2) plane for IGW. Typical triads, scattered in the regions defined by (4.3), are shown in figures 1 (BRW) and 3 (IGW). The value of K_a and the frequencies (wavenumbers) are shown in figure 2 (figure 4) for BRW (IGW). All variables have been scaled using K_3 and β (BRW) or N (IGW).

The interaction, (3.11), and resonance, (4.1), conditions are easily satisfied in the following way.

(*a*) For BRW it is $\alpha_a = \pi + \theta_b - \theta_c$ ($abc = 123 + cp$) in order to satisfy (3.11), and the resonance condition then reduces to

$$\tan\theta_3 = [\cos 2\alpha_3 \cos(\alpha_1 - \alpha_2) + \cos\alpha_3]/[2\sin(\alpha_1 - \alpha_2)\sin^2\alpha_3].$$

(*b*) For IGW, given the three frequencies it is $\theta_a = l_a \cos^{-1}(\omega_a/N)$ (with $l_a^2 = 1$), and, in order to satisfy (3.11), it is $K_a \doteq \sin(\theta_b - \theta_c)$ ($abc = 123 + cp$). The four classes of triads shown correspond to different choices of $[l_1, l_2, l_3]$, namely I: $[+--]$, II: $[+-+]$, III: $[--+]$ and IV: $[---]$. For classes I and II the three waves propagate in the same horizontal direction, and are therefore similar to BRW in the sense that the component with largest $|\omega|$ has intermediate $|\mathbf{k}|$. For classes III and IV, on the other hand, this is not true because the component with smallest $|\omega|$ propagates in a direction opposite to the other two.

'Most' triads in figures 1–4 have comparable wavenumbers and frequencies. However there are non-local triads, in K or ω space or both, that fall into three categories, namely

$$|\mathbf{k}_a| \ll |\mathbf{k}_b| \simeq |\mathbf{k}_c| \begin{cases} \mathbf{s}_b = -\mathbf{s}_c, \quad \omega_b = \omega_c = -\tfrac{1}{2}\omega_a \quad \text{(IGW; } a = 3\text{)}, & (4.4a) \\ \mathbf{s}_b = \mathbf{s}_c, \quad \mathbf{s}_a . \partial\omega_a/\partial\mathbf{k}_a = 1 \quad \text{(BRW, IGW; } a = 1\text{)}, & (4.4b) \end{cases}$$

$$|\omega_a| \ll |\omega_b| \simeq |\omega_c| \quad \mathbf{s}_b = \mathbf{s}_c - 2\hat{\mathbf{k}}_a \mathbf{s}_c . \hat{\mathbf{k}}_a \quad \text{(BRW, IGW; } a = 1\text{)}. \tag{4.4c}$$

The limit (*a*), at the lower right boundary of classes III and IV of IGW, is the trapeze instability (or parametric instability) region predicted by Orlanski (1973).

In the limit (*b*), at the leftmost boundary of BRW and classes II and III of IGW, the resonance condition expresses that the most efficient interactions are among short and fast wave packets travelling along a constant phase line of a long and slower wave. For IGW, this is the region of induced diffusion of the McComas & Bretherton (1977) analysis.

Finally, in the limit (c), at the rightmost boundaries for BRW and leftmost boundary for IGW of classes II and III, waves b and c differ only by a reflexion on a plane perpendicular to $\mathbf{k}_a$. Wave a is quasi steady and its wavenumber is twice the wavenumber of the other two in the direction of $\mathbf{k}_a$. For the case of IGW, this is the elastic scattering region of the analysis of McComas & Bretherton (1977).

The scaled interaction coefficient is everywhere of order unity, i.e. (see table 2)

$$|\Gamma| \sim |s_3| \quad (\text{IGW}; \text{BRW}, K_R = 0), \tag{4.5}$$

The $\Gamma = 0$ values in the figures correspond to $s_1 = s_2 = s_3$, for BRW and IGW of classes I and II, or to parallel $\mathbf{k}$'s for IGW of class IV. Note that use of $|\mathbf{k}_1|$ or $|\mathbf{k}_2|$, instead of $|\mathbf{k}_3|$, to scale Γ would have produced a misleading value of zero in the limit (4.4a), or a value of infinity in the limit (4.4b). Thus, the scale for the interaction coefficient is given by the intermediate x-slowness.

The interaction coefficient for BRW much larger than the deformation radius is considerably smaller, namely

$$|\Gamma| \sim |s_3|(K_3/K_R)^2 \quad (\text{BRW}, K_a \ll K_R) \tag{4.5}'$$

as can be shown from the formulae in table 2.

Equations (4.5) and (4.5)′ somewhat simplify the picture of nonlinear interactions in the sense that most of the information on the parameters of the resonant terms in (3.9) is contained in the dispersion relation. Moreover, (4.5) and (4.5)′ can be used to estimate the order of magnitude of the coupling coefficients of *any* triad (resonant or off-resonant), in the following way:

Given three components that satisfy the interaction condition (3.11) (but not necessarily the resonance condition (4.1)), let 3 be the one with intermediate value of s (but not necessarily with the largest $|\omega|$). The three coupling coefficients associated with this triad are invariant under the rotation $\theta_a \to \theta_a' = \theta_a + \delta\theta$ (see table 2) because (3.11) requires $K_a \doteqdot \sin(\theta_b - \theta_c)$, $abc = 123 + cp$, and there are two values of $\delta\theta$ (that differ by π) such that the rotated components are in resonance. Equations (4.5) and (4.2) then imply $\sigma_3^{12} \simeq K_3 \cos\theta_3'$, and furthermore it is $|\cos\theta_3'| > |\cos\theta_{1,2}'|$. Consequently, it is

$$\left.\begin{aligned} |\sigma_3^{12}| &\simeq |K_3| \quad (\text{IGW}; \text{BRW}, K_R = 0), \\ |\sigma_3^{12}| &\simeq K_3^4/K_R^3 \quad (\text{BRW}, K_a \ll K_R). \end{aligned}\right\} \tag{4.6}$$

That is, given any triad (resonant or not) the coupling coefficient corresponding to the component with intermediate x-slowness is of the order of magnitude of its total wavenumber. The other two coupling coefficients are given by (3.15) (see also (3.17)).

5. One-wave and three-wave problems

In this section the formalism developed in § 3 is used to study two simple problems: one-wave (and its linear stability analysis) and three-wave systems. One single wave was shown to be an exact nonlinear solution. To study its stability, an infinite number of other components are considered, but (3.9) is linearized in this perturbation, violating the conservation laws. For the three-wave problem, on the other hand, (3.7) is truncated to the minimum number of (physical) components which exhibit non-

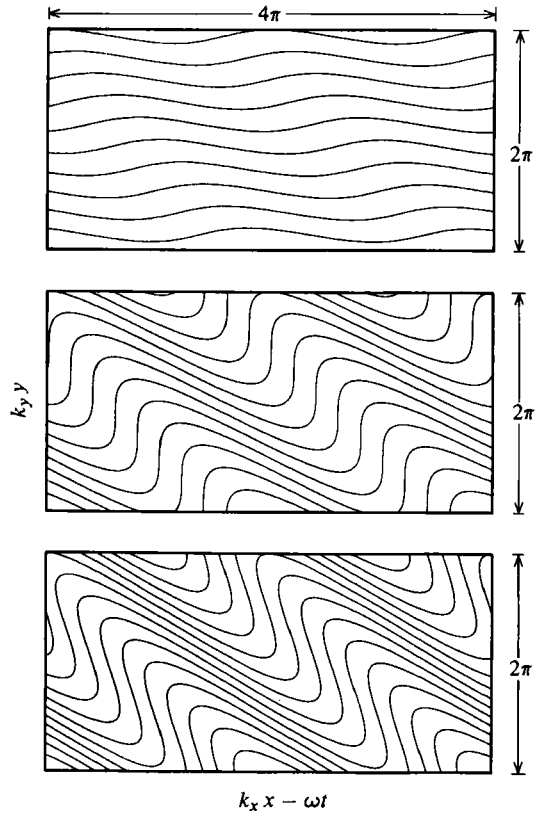

FIGURE 5. Isolines of potential vorticity (total density) for one Rossby wave (internal gravity wave). As the pattern moves in the x direction, the fluid particles stay in one of the isolines by means of displacements perpendicular to $\mathbf{k}$; i.e. the trajectories are straight lines (with a slope $= -1$ in the co-ordinates of the figure). The top, middle and bottom graphs correspond to max $(su) = 0{\cdot}2$, $1{\cdot}0$ and $1{\cdot}5$, where u and $1/s$ are the particle velocity and phase speed in the x-direction.

linear behaviour, but all the corresponding nonlinear terms in (3.9) are kept (which was already shown to preserve conservation of E and P).

Properties of one wave

One single wave, for example,

$$Z_a(t) = Z_a(0)\exp(-i\omega_a(t)), \quad Z_b(t) = 0 \quad \text{for} \quad b \neq a, \tag{5.1}$$

is an exact solution of (3.9) because any coupling coefficient of the form σ_b^{aa} and σ_b^{aa*} vanishes, owing to the sum rules (3.15).

A non-dimensional amplitude of the wave can be defined as the ratio of the maximum particle speed to the phase speed along x, namely†

$$M = |s_a|\, V = \begin{cases} 2K_a(K_a^2 + K_R^2)^{\frac{1}{2}} E^{\frac{1}{2}}/\beta & \text{(BRW)}, \quad (5.2a) \\ (2K_a^2 E)^{\frac{1}{2}}/N & \text{(IGW)}, \quad (5.2b) \end{cases}$$

where V is the maximum value of the speed $|\mathbf{v}|$, produced by the wave.

† For non-divergent BRW ($K_R = 0$) this coincides with the quantity M used by Gill (1974).

The isolines of potential vorticity (BRW) or total density (IGW) are shown in figure 5 for different values of $M\,|\sin\theta|$ (i.e. the ratio of the maximum particle x-velocity over phase speed along x).

The average momentum vanishes, whereas the *pseudo-momentum* does not. In fact, the particle positions are given *exactly* by the equation

$$\mathbf{x} = \mathbf{x}' + M|\mathbf{k}|^{-1}(\tan\theta, 1)\cos(\mathbf{k}.\mathbf{x}' - \omega t), \tag{5.3}$$

(where $\mathbf{x}'$ is the equilibrium position), which shows that, unlike the irrotational case, there is no mean drift. This is then another example of the difference (already pointed out by Andrews & McIntyre 1978) between momentum and pseudo-momentum.

One single wave is a rather trivial solution of (3.9), because the nonlinear terms vanish for all values of M. The simplest nonlinear problem one can study is the stability of that wave, i.e., writing

$$\phi = (Z_a\phi_a + \phi') + *, \tag{5.4}$$

with $E'(0) \ll E_a(0)$, to consider the question whether component 'a' might release a finite amount of its energy to other components. A *necessary condition*, due to the conservation of P/E, is that the wave expansion of ϕ' must contain components with values of s both larger and smaller than s_a (see point 1 after (3.15)).

The usual procedure (Gill 1974) is to linearize (3.9) in ϕ' and seek eigensolutions such that $E'(t) = E'(0)\,|\exp(2\mu t)|$. The 'perturbation' ϕ' is then formed by all components whose wavenumbers differ by a multiple of $\mathbf{k}_a$, and the growth rate μ is found by solving for the root of an infinite-order determinant. Approximating this root by that of a 2×2 determinant, we find that

$$\mu^2 \simeq \sigma_b^{ca}\sigma_c^{ab}E_a - \tfrac{1}{2}(\omega_a+\omega_b+\omega_c)^2; \tag{5.5}$$

corresponding to wave 'a' interacting only with waves 'b' and 'c', with the three wavenumbers related by (3.11). Although (5.5) might not be a good approximation for the growth rate, it illustrates three important points. (a) For instability ($\mu^2 > 0$) it is necessary that the product of the two coupling coefficients be positive, i.e. that s_a be between s_b and s_c (see (3.16)); already mentioned to be a consequence of the conservation of E and P. (b) For a given eigenfunction, there might be a threshold of E_a for instability, as suggested by the second term on the right-hand side of (5.5). (c) The maximum growth rate is proportional to $E_a^{\frac{1}{2}}$.

In the *weak-interactions limit* ($E_a \to 0$), (5.5) is asymptotically correct (Gill 1974), and wave 'a' is seen to be unstable for any amplitude if it belongs to a resonant triad and point (a) of the last paragraph is satisfied, which was shown in the last section to imply $|\omega_a| \geqslant |\omega_{b,c}|$. It is clear then that all BRW or IGW with $\omega \neq 0$ are unstable, even in this limit, because they are the intermediate x-slowness component of many resonant triads. Use of (4.5) in (5.5) shows that the maximum growth rate is proportional to the wave amplitude M and frequency ω. In fact, taking the maximum of (5.5) over all resonant triads, the maximum growth rates corresponding to the extreme orientations of wave 'a' are found to be

$$\mu_{\max} = \begin{cases} 0{\cdot}3248\,M|\omega_a| & \text{(IGW)}, \qquad (5.6a) \\ 0{\cdot}1617\,M|\omega_a| & \text{(BRW, } K_R = 0), \qquad (5.6b) \\ 0{\cdot}0833\,M|\omega_a|\,K_a^2/K_R^2 & \text{(BRW, } K_R \gg K_a), \qquad (5.6c) \end{cases}$$

for $[E_a \to 0, \theta_a = 0]$ and

$$\mu_{\max} = \begin{cases} 0{\cdot}3936\, M|\omega_a| & (\text{IGW}), \quad (5.7a) \\ 0{\cdot}4496\, M|\omega_a| & (\text{BRW}, K_R = 0), \quad (5.7b) \\ 0{\cdot}75\, M|\omega_a|\, K_a^2/K_R^2 & (\text{BRW}, K_R \gg K_a), \quad (5.7c) \end{cases}$$

for $[E_a \to 0, \theta_a \to \pi/2]$. Thus, the non-dimensional amplitude, defined by (5.2), is approximately equal to the ratio of the linear, $|2\pi/\omega|$, to the nonlinear, $(1/\mu)$, time scales.

For very long BRW, though, M is of the order of the ratio between μ and $\omega_a K_a^2/K_R^2$, instead of ω_a. But these waves are almost non-diversive (with a zonal phase speed approximately equal to $-\beta/K_R^2$), and $\omega_a K_a^2/K_R^2$ is precisely the dispersive part of the frequency. For quasi-non-dispersive BRW, then, the relevant linear time scale is the (Doppler shifted) period measured by an observer that moves with velocity $-\beta/K_R^2$ along x.

The quantity $I = \langle(\Delta\psi/N)^2 + (\nabla\eta)^2\rangle/2$ that Milder (1976) showed to be conserved in the *linear* IGW problem (already mentioned in § 2) has the wave-expansion representation $I = \Sigma' s_a^2 E_a^2$ (for a single wave it is $I = M^2/2$). However, it is easy to show (using its wave representation and (3.14)) that I is *not* conserved in the nonlinear problem, even allowing for only resonant interactions (on the other hand, *any* function of the E_a is an 'integral of motion' in the linear problem). For instance, for the system of a wave decaying by means of the trapeze instability mechanism (see (4.4*a*)), the value of I increases indefinitely with time, even though the system may be tending to a saturation state (Orlanski & Cerasoli 1980).

In the *strong-interactions limit* $(E_a \to \infty)$, on the other hand, the linear terms in (3.9) are negligible, and therefore the values of μ are independent of θ_a (because the coupling coefficients of each triad depend only on the relative orientations of the three $\mathbf{k}$'s). The approximation (5.5) is no longer valid in this limit, and the values of μ can be calculated solving for the roots of the infinite determinant (see Gill 1974) in an iterative way. The maximum growth rates are found to be

$$\mu_{\max} = \begin{cases} 0{\cdot}2697\, |\mathbf{k}_a|\, V & (\text{IGW}), \quad (5.8a) \\ 0{\cdot}2697\, |\mathbf{k}_a|\, V & (\text{BRW}, K_R = 0), \quad (5.8b) \\ 0{\cdot}1833\, |\mathbf{k}_a|\, V K_a^2/K_R^2 & (\text{BRW}, K_R \gg K_a), \quad (5.8c) \end{cases}$$

for $[E_a \to \infty$, any $\theta_a]$, where V, as before, is the maximum of $|\mathbf{v}|$. For comparison with (5.6) and (5.7), note that (5.2) implies

$$M|\omega_a| = |\mathbf{k}_a|\, V\, |\cos\theta_a|. \quad (5.9)$$

The weak- and strong-interaction limits correspond mathematically to $M \to 0$ and $M \to \infty$. However, from a physical point of view, one would like to have an idea about when some value of M can be considered 'small' or 'large': some boundary in (M, θ) space is needed to separate weak and strong decay. A good candidate is

$$M\, |\sin\theta| = 1,$$

for the following reason.

For $M\, |\sin\theta| < 1$ (e.g. the top panel of figure 5) the restoring mechanism, i.e. the meridional (vertical) gradient of potential vorticity (total density), for the case of BRW (IGW), has the same sign everywhere. The effect of the wave on the perturbation

is to alter its natural frequency by an amount of $O(M)$ (which, in particular, can be complex, leading to instability of the wave). For $M|\sin\theta| > 1$ (e.g. the lower panel in figure 5), on the other hand, the perturbation does not 'feel' a restoring mechanism everywhere in the fluid. The value $M|\sin\theta| = 1$ corresponds to the onset of inertial (gravitational) instability of BRW (IGW); and also coincides with Orlanski's (1971) criteria that the maximum particle x-velocity equals the phase speed in that direction, since

$$M|\sin\theta| = \max(su). \tag{5.10}$$

Furthermore, a $\omega = 0$ 'wave' (i.e. the limit of the expressions in table 2 when $\cos\theta \to 0$) cannot be unstable in the weak-interactions limit, because conservation of E and P would require it to be the member of a resonant triad with maximum $|\omega|$, which is clearly absurd. It is easy to show that this 'wave' is unstable only for $M > 1$.

The three-wave problem

The problem of the stability of a single wave gives the nonlinear time scale $(1/\mu)$ and the properties of the energy exchange among the components of an hypothetical system with most of the energy initially in one wave. The solution, though, is clearly not valid after a time t such that $E'(0)\exp(2\mu t) \simeq E_a(0)$ (or $|P'(0)|\exp(2\mu t) \simeq |P_a(0)|$, whichever happens first), because the interaction among the components of ϕ' becomes important. A clear indication of this is that linearization in ϕ' violates conservation of E and P.

For a 'simplified' system obtained by an arbitrary truncation of (3.7) to N (physical) components, however, E and P are conserved if all nonlinear terms involving these components are kept in (3.9). The simplest case is obtained for $N = 3$ (Bretherton 1964), namely

$$\dot{Z}_i + i\omega_i Z_i = \sigma_i Z_j^* Z_k^*, \tag{5.11}$$

where (from now on) $ijk = 123 +$ cyclic permutations, and the upper indices of the coupling coefficients are omitted for simplicity.

Equation (5.11) is obviously an extreme simplification of (3.9) and can only be justified for a finite time and for problems in which initially most of the energy is in a resonant (*or quasi resonant*) triad† (Bretherton 1964). Nevertheless, a particular solution of (5.11) found by McGoldrick (1965) (equation (5.23) below) is often quoted in the literature, because even a very simple problem like (5.11) might illustrate part of the dynamics of more complicated systems. Furthermore, a system like (5.11) may be found in quite different nonlinear problems of geophysical fluids. For instance, Craik & Adam (1979) use (5.11) with a resonant triad of capillary–gravity waves in a three-layer fluid in order to explain the nonlinear 'explosive' instability of a system which is linearly stable.

The general solution of (5.11) (for any initial condition and for both the resonant and off-resonant cases) is reported here. Its properties are discussed for problems where the coupling coefficients satisfy (3.15), or more generally, where all the σ's do not have the same sign. However, the general solution of (5.11) for problems like that

† It is also necessary for the pair formed by one of the three components and the conjugate of one of the other two not to be in resonance with a fourth wave. This is true for almost every triad in figures 1–4.

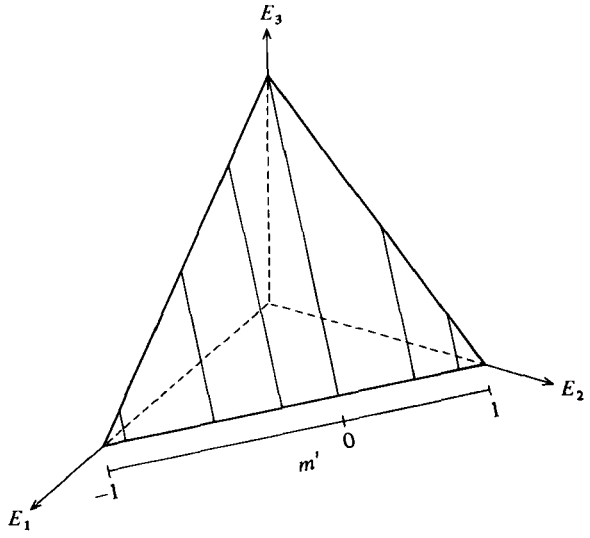

FIGURE 6. Evolution of the three-wave system in energy space. Conservation of total energy requires the system to move over the triangle. Conservation of total x-pseudomomentum constrains the evolution to a straight line in that triangle; several of those lines are drawn and labelled by the parameter m'. The third conservation law (5.15) usually restricts the trajectory to be a shorter segment of that line. Component 3 is the one with intermediate x-slowness; for a resonant trio this is also the one with maximum $|\omega|$.

of Craik & Adams (1979) (which corresponds to the three σ's with the same sign) is the one reported in this section, in a different range of the parameters.

The system (5.11) can be rewritten as

$$\dot{E}_i = 2\sigma_i (E_1 E_2 E_3)^{\frac{1}{2}} \cos\delta, \tag{5.12a}$$

$$\omega_i' = \omega_i - \sigma_i (E_j E_k / E_i)^{\frac{1}{2}} \sin\delta, \tag{5.12b}$$

where $\delta = \arg(Z_1 Z_2 Z_3)$ is the relative phase and $\omega_i' = -d(\arg Z_i)/dt$ are the instantaneous frequencies. Periods when the relative phase is close to 0 or π correspond to maximum energy exchange and minimum departure from the 'free' frequencies; whereas the converse is true when the relative phase is close to $\pm\frac{1}{2}\pi$. The solution found by McGoldrick (1965) corresponds to an initial condition such that $\sin\delta$ or one of the energies vanishes, and is only valid in the strictly resonant case, i.e. for $\Omega = 0$, where

$$\Omega = \omega_1 + \omega_2 + \omega_3 \tag{5.13}$$

is the detuning of the trio.

The system (5.11) has three integrals of motion. On the one hand, equations (3.15) imply

$$E = E_1 + E_2 + E_3 = \text{constant}, \tag{5.14a}$$

$$P = s_1 E_1 + s_2 E_2 + s_3 E_3 = \text{constant}, \tag{5.14b}$$

or, more generally,

$$A_i - A_j = \text{constant}, \tag{5.14)'}$$

where $A_i = E_i \sigma_j \sigma_k$. For, and only for, a resonant trio, (4.2) implies that the A_i are proportional to the 'wave-actions' E_i/ω_i.

On the other hand, it is easy to show that (Bretherton 1964)

$$S = \sigma_1\sigma_2\sigma_3(E_1E_2E_3)^{\frac{1}{2}}\sin\delta + \Omega\Sigma A_i/6 = \text{constant}. \tag{5.15}$$

The physical consequences of the three conservation laws can be seen as successive constrains on the energy exchange.

(i) (5.14*a*) implies that the system evolves in a triangle in energy space (see figure 6).

(ii) (5.14*b*) defines a straight line in that triangle. This line is labelled by the parameter

$$m' = 2(A_1 - A_2)/(2A_3 - A_1 - A_2 + |A_1 - A_2|) \tag{5.16}$$

that ranges from -1 to 1. The line is infinitesimally small if most of the energy is in component 1 ($m' \simeq -1$) or 2 ($m' \simeq 1$), and the largest of those lines ($m' = 0$) goes from $[E_1 = E_2 = 0, E_3 \neq 0]$ to $[E_1\sigma_2 = E_2\sigma_1, E_3 = 0]$. The components are numbered so that $\text{sign}(\sigma_1) = \text{sign}(\sigma_2) = -\text{sign}(\sigma_3)$ (which, with (3.15), means that s_3 is between s_1 and s_2).

(iii) If $\Omega = 0$ and $\sin\delta \neq 0$ at $t = 0$, (5.15) constrains even further the evolution along this line (because $|\sin\delta| \leqslant 1$), and the three phases are somewhat locked (because $\sin\delta$ cannot change sign). If $\Omega \neq 0$ and the initial energies are small enough so that the first term in the right-hand side of (5.15) is negligible, then $E_i(t) \simeq E_i(0)$ (i.e., the detuning inhibits energy exchange), because (5.14) and (5.15) represent three constraints on the three energies.

The integrals of motion can be used to define two other parameters that, together with m', fully characterize the solution: a nonlinear inverse time scale μ_0 given by (A 2*a*), and another non-dimensional parameter m given by (A 4*b*). The parameter μ_0 is a function of E and m', and satisfies

$$|\mu_0| \lesssim |\sigma_3 E^{\frac{1}{2}}|. \tag{5.17}$$

The parameter m, which is a measure of the energy exchange (see (5.19) below), is a function of E, m' and S; and it has, roughly, smaller values for larger values of $|S|$. The range of m is

$$0 \leqslant m \leqslant 1 - |m'|. \tag{5.18}$$

The solutions of (5.12) take the form (see appendix)

$$E_i(t) = \mu^2(m\,\text{sn}^2\mu t - \lambda_i)/\sigma_j\sigma_k, \tag{5.19}$$

where sn(...)is the Jacobian elliptic sine (Abramowitz & Stegun 1966, chapter 16) with parameter m, and the parameters μ and λ_i can be easily calculated from any initial condition (see appendix). For $m < 1$, $\text{sn}^2(u)$ is a periodic function of u, which varies between 0 (for $u = 0, \pm K(m), \pm 2K(m), \ldots$) and 1 (for $u = \pm K(m), \pm 3K(m), \ldots$). $K(m)$ is the elliptic integral of the second kind, which is equal to $\pi/2$ at $m = 0$ and slowly tends to infinite as $m \to 1$. In particular it is $\text{sn}(\ldots) = \sin(\ldots)$ for $m = 0$ and $\text{sn}(\ldots) = \tanh(\ldots)$ for $m = 1$.

Finally, replacing (5.15), (5.19) and (A 2*c*) in (5.12*b*), the 'instantaneous frequencies' are found to be of the form

$$\omega_i' = \omega_i - \Omega/2 + \mu B_i/(m\,\text{sn}^2\mu t - \lambda_i), \tag{5.20}$$

where the B_i are constants of order one. The last two terms on the right-hand side can be decomposed into a mean value (frequency shift) plus a periodic fluctuation

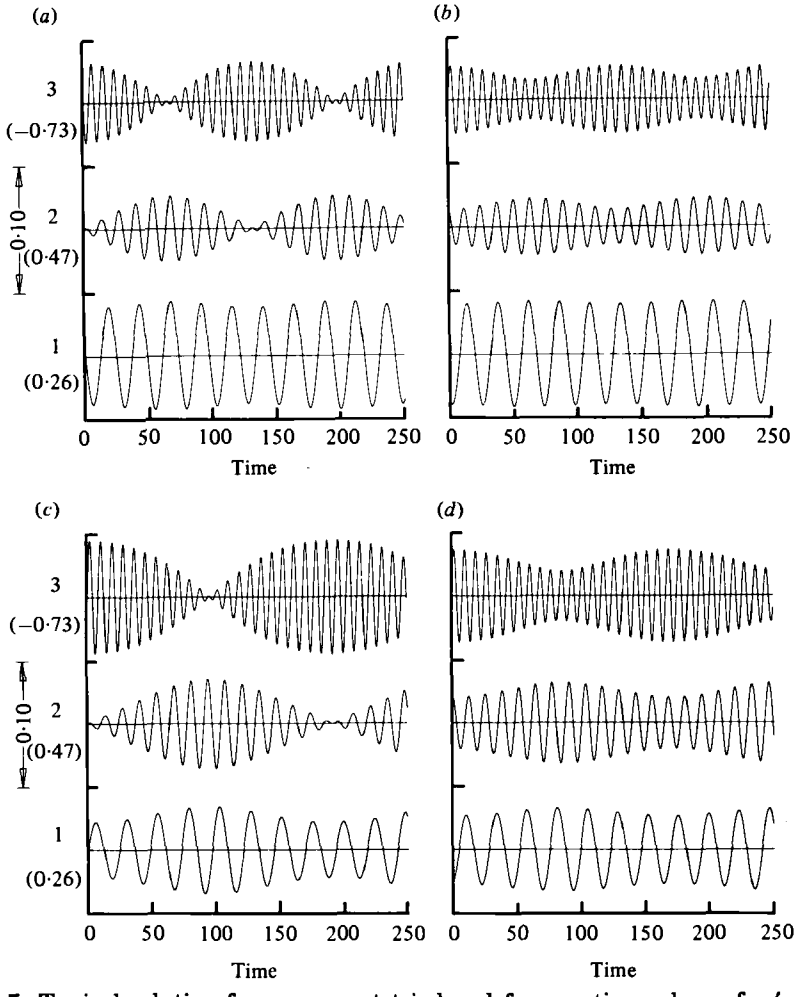

FIGURE 7. Typical solution for a resonant triad and for negative values of m'. Real $(z_i)/v$ is plotted against time. The free frequency of each component is shown between parenthesis on the left. (*a*) $\mu = 0{\cdot}025$, $m' = -0{\cdot}80$, $m = 0{\cdot}20$; (*b*) $\mu = 0{\cdot}025$, $m' = -0{\cdot}80$, $m = 0{\cdot}10$; (*c*) $\mu = 0{\cdot}020$, $m' = -0{\cdot}40$, $m = 0{\cdot}60$; (*d*) $\mu = 0{\cdot}020$, $m' = -0{\cdot}40$, $m = 0{\cdot}29$.

(frequency modulation). This equation can also be integrated in terms of the elliptic integral of the third kind (Abramowitz & Stegun 1966, chapter 17) in order to find the evolution of the individual phases.

The properties of all possible solutions in the *resonant* case ($\Omega = 0$) are presented now, leaving the discussion of *detuning effects* to the end of this section.

Solution of the resonant case

Typical solutions of (5.11) are shown in figures 7–9 for different values of m' and m, scattered in the region of (5.18). The three frequencies are equal to 0·26, 0·47 and $-0{\cdot}73$, the value of μ is, in all cases, approximately equal to 0·02 and the elapsed time is equal to 250 in the inverse of the (arbitrary) unit of the frequencies. Only the real part of the amplitudes Z_i is shown, and it is scaled with $v = 1/\Gamma$ (see (4.2)).

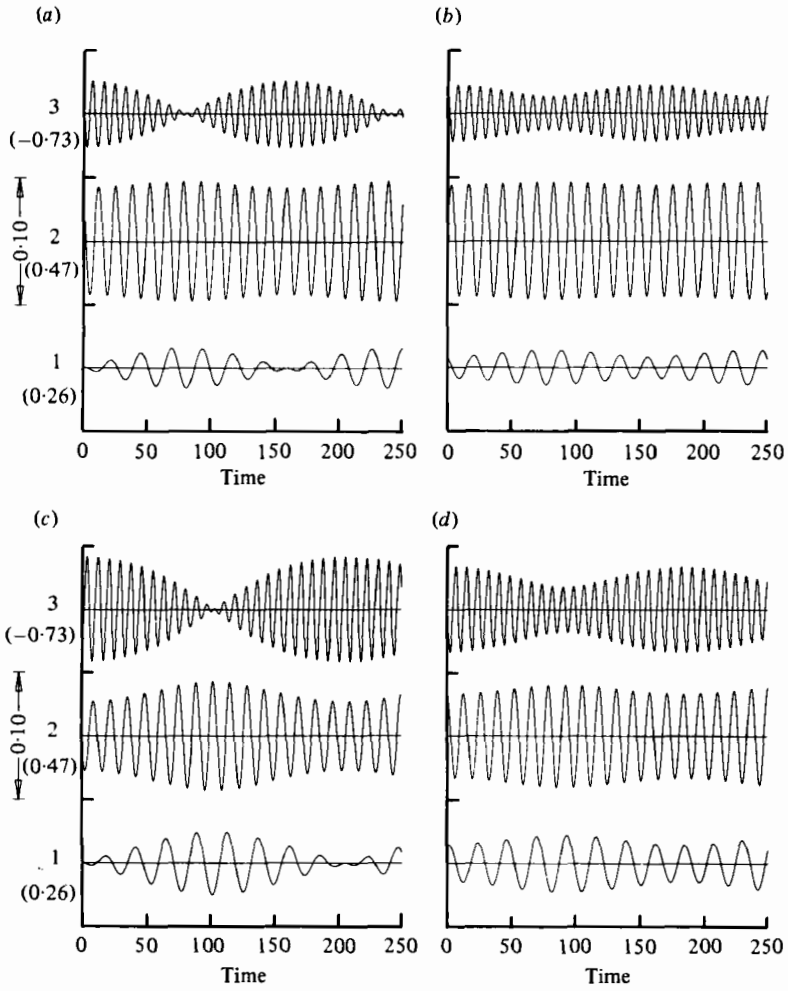

FIGURE 8. As in figure 7 for positive m'. (a) $\mu = 0{\cdot}021; m' = 0{\cdot}80, m = 0{\cdot}20$; (b) $\mu = 0{\cdot}020, m' = 0{\cdot}80, m = 0{\cdot}10$; (c) $\mu = 0{\cdot}019, m' = 0{\cdot}40, m = 0{\cdot}60$; (d) $\mu = 0{\cdot}019, m' = 0{\cdot}40, m = 0{\cdot}29$.

The parameters of the examples of figures 7–9 are shown as small circles in the $[m, m']$ plane in figure 10. Also in that figure are indicated the properties of the solution at the boundaries ($m = 0$ or $1 - |m'|$) of the parameter space, which are the following.

(a) For $m = 0$ (bottom boundary of the triangle in figure 10; this corresponds to a maximum value of $|S|$, given E and m') the three energies are constant, (5.19), and the three frequencies are also constant but shifted from their free values by an amount of $O(\mu_0)$. This solution is obtained for (see (5.12a)) $\sin\delta = \pm 1$ and, to preserve the value of δ, for (see (5.12b)) $\Sigma\sigma_i/E_i = 0$, which implies (see (3.15))

$$E_i = E_0/(1 + U_0 s_i), \tag{5.21}$$

where E_0 and U_0 are constants. The shifted frequencies are given by

$$\omega_i' = (1 \pm \epsilon)\,\omega_i \pm \epsilon k_i U_0, \tag{5.22}$$

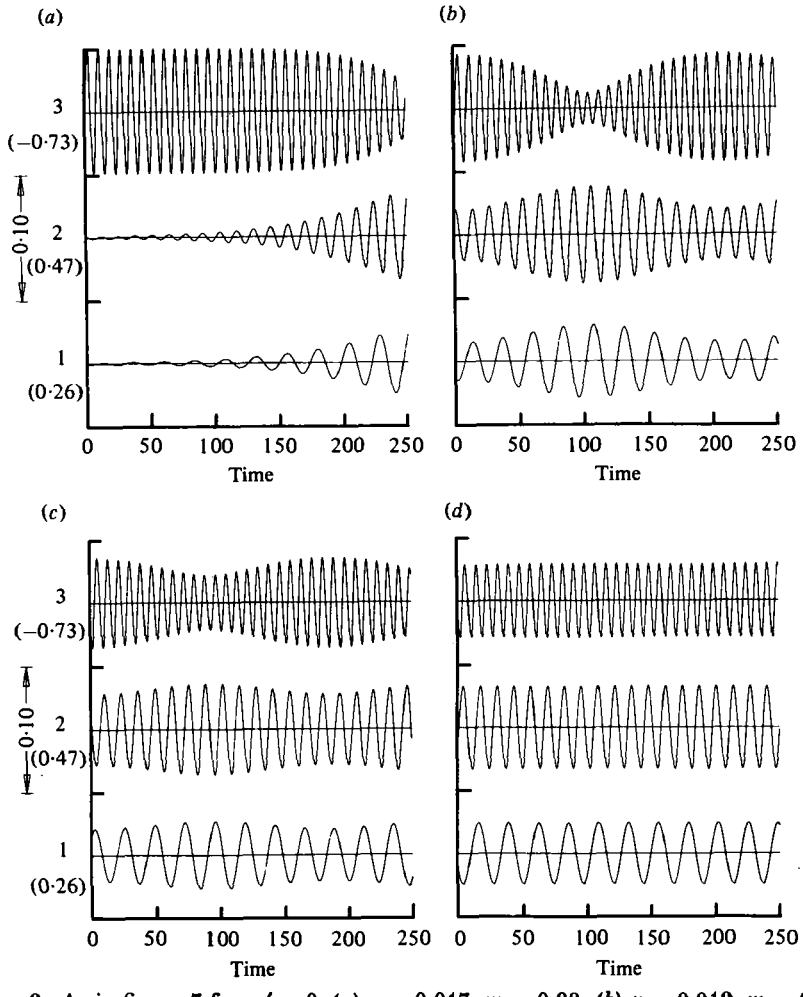

FIGURE 9. As in figure 7 for $m' = 0$. (*a*) $\mu = 0{\cdot}017$, $m = 0{\cdot}99$; (*b*) $\mu = 0{\cdot}019$, $m = 0{\cdot}59$; (*c*) $\mu = 0{\cdot}018$, $m = 0{\cdot}29$; (*d*) $\mu = 0{\cdot}018$, $m = 0{\cdot}02$.

(with $\epsilon = \Gamma(E_1E_2E_3)^{\frac{1}{2}}/E_0$) and satisfy $\Sigma\omega_i' = 0$. The example in the right bottom graph of figure 9 ($m' = 0, m = 0{\cdot}02$) corresponds very closely to this case (since the maximum value of m, for $m' = 0$, is 1).

Incidentally, (5.21) also gives the equilibrium solution of the Boltzmann equation for stochastic systems (Holloway & Hendershott 1977).

(*b*) For $m = 1 - |m'|$ (upper boundaries of the triangle in figure 10; this corresponds to $S = 0$), on the other hand, the solutions experience the maximum energy exchange compatible with conservation of E and P (for example, for $m < 1$, two of the energies vanish alternatively) while the frequencies remain constant at their free values, and the relative phase is such that $\sin\delta = 0$. The examples corresponding to this other extreme case are the two to the left in figures 7 and 8 and the one at the top left in figure 9.

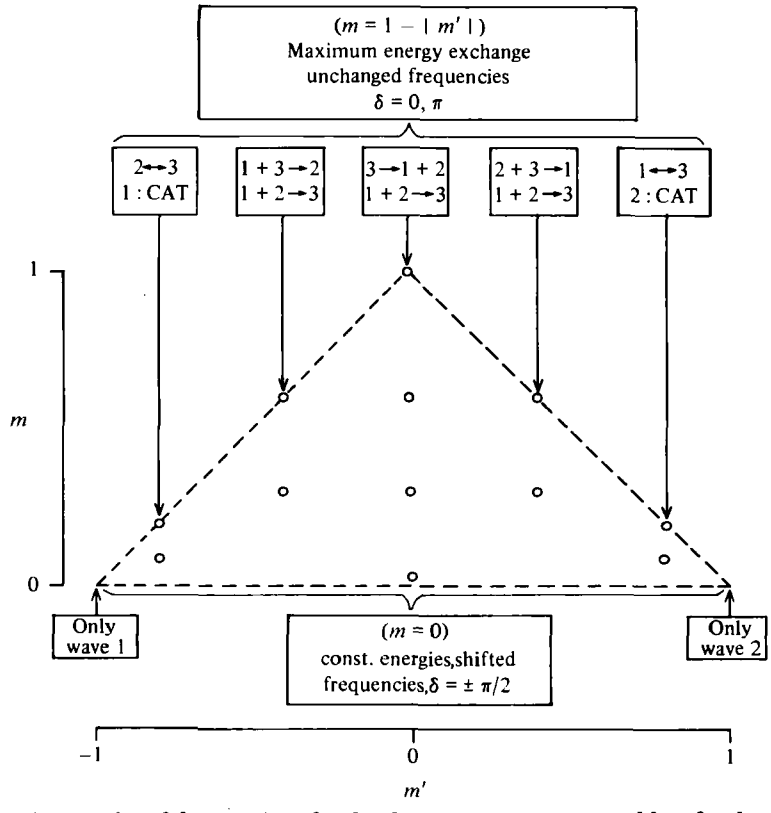

FIGURE 10. Properties of the solutions for the three-wave resonant problem for the extreme values (0 and $1-|m'|$) of the parameter m. Any point in the interior of the triangle represents also a possible solution. The values of (m, m') for the examples of figures 7–9 are shown as small circles.

For $m' < 0$, the parameters of the solution are $\lambda_1 = 1$, $\lambda_2 = m$, $\lambda_3 = 0$ and

$$\mu = \sigma_3[E\omega_1\omega_2/(\omega_1^2 + m\omega_2^2 + (1+m)\,\omega_1\omega_2)]^{\frac{1}{2}}.$$

The amplitudes are given by

$$Z_1 = [\omega_1 E/(\omega_1 + m\omega_2)]^{\frac{1}{2}}\,\mathrm{dn}(\mu t)\exp(-i\omega_1 t), \tag{5.23a}$$

$$Z_2 = [\omega_2 mE/(\omega_1 + m\omega_2)]^{\frac{1}{2}}\,\mathrm{cn}(\mu t)\exp(-i\omega t), \tag{5.23b}$$

$$Z_3 = [-\omega_3 mE/(\omega_1 + m\omega_2)]^{\frac{1}{2}}\,\mathrm{sn}(\mu t)\exp(-i\omega_3 t), \tag{5.23c}$$

times any constant phase factors (constrained by $\sin\delta = 0$). This particular solution is the one found by McGoldrick (1965) and is often quoted in the literature.

For $m' \simeq 1$ ($|A_1| \gg |A_{2,3}|$), $m = 1 + m'$, there is a catalytic exchange of energy between components 2 and 3, ($2 \Leftrightarrow 3$), in the sense that E_1 remains essentially constant. The presence of component 1, however, is very important in determining the nonlinear time scale, since in this limit it is $\mu^2 \simeq \sigma_2\sigma_3 E_1$. This case is shown in the top left of figure 7. For a system such that $\omega_1 = 0$ it is always $m' = -1$ and $m = 0$, i.e. this is the only possible solution of the problem.

For a value of m' between -1 and 0 (bottom left of figure 7) E_1 no longer stays constant. At $t = 0$ the process looks like the resonant generation of 2 by the interaction of 1 and 3, $(1+3 \to 2)$; then half an energy period later ($t = K(0{\cdot}6)/\mu \simeq 97$) the process looks like $(1+2 \to 3)$. There is not a clear distinction between this case and the one discussed before, as in fact the properties of the solution vary continuously throughout parameter space.

The examples with $m' > 0$ (graphs on the left of figure 8) are like the ones just discussed, but the roles of components 1 and 2 are interchanged.

Finally, for $m' \simeq 0$ and $m \simeq 1$ (top left of figure 9) the process looks like the decay of the unstable component, $(3 \to 1+2)$, followed by its resonant regeneration, $(1+2 \to 3)$; and so on. Note the increase of the nonlinear energy period as $m \to 1$. For m strictly equal to 1 ((5.26) below, with $\Omega = 0$) this period tends to infinity and the whole process (from $t = -\infty$ to $+\infty$) is $(3 \to 1+2)$ followed by $(1+2 \to 3)$, just once.

(*c*) For an intermediate value of m, between 0 and $1-|m'|$ (interior of the triangle in figure 10; this corresponds to an intermediate value of $|S|$), none of the energies ever vanish, and the instantaneous frequencies are not constant (i.e. there is a frequency modulation). The examples corresponding to this case are the two graphs on the right in figures 7 and 8, and the top right and bottom left in figure 9.

Detuning effects

The main effect of the detuning, $\Omega \neq 0$, is to inhibit the energy exchange. Its importance is measured by the magnitude of

$$\Omega' = \Omega/2\mu_0. \tag{5.24}$$

As $|\Omega'| \to \infty$ (very low energies) it is $m \sim O(\Omega'^{-3})$, or $m \sim O(\Omega'^{-4})$ if one of the energies initially vanishes, and $|\mu| \sim \frac{1}{2}|\Omega|$. The three energies remain essentially constant and the instantaneous frequencies are shifted by an amount of $O(\mu_0)$. The value $|\Omega'| = 1$ is roughly the boundary between resonant and off-resonant regimes.

For instance, if initially $E_3 \gg E_{1,2}$ (near the apex in figure 6) then it is

$$m' = 0, \quad \mu_0^2 = \sigma_1\sigma_2 E_3, \quad \mu = \mu_0|1-\Omega'^2|^{\frac{1}{2}}$$

and

$$m = \begin{cases} 1 & \text{if} \quad |\Omega'| < 1 \\ 0 & \text{of} \quad |\Omega'| > 1. \end{cases} \tag{5.25}$$

Therefore, component 3 will decay into components 1 and 2 only if its energy is high enough so that $|\Omega'| < 1$. In this case, the amplitudes are given by

$$\left.\begin{aligned} Z_i &= [(\Omega'^2-1)\,E\sigma_i/\sigma_3]^{\frac{1}{2}}\,\mathrm{sech}\,(\mu t)\exp[-i(\omega_i - \tfrac{1}{2}\Omega)\,t] \quad (i = 1, 2), \\ Z_3 &= E[(1-\Omega'^2)^{\frac{1}{2}}\tanh(\mu t) + i\Omega']\exp(-i\omega_3 t). \end{aligned}\right\} \tag{5.26}$$

Note that component 3 releases energy to components 1 and 2 (for $-\infty < t < 0$) and then gains it back (for $0 < t < \infty$). However, unlike the resonant case, component 3 does not loose all of its energy; its minimum energy being $\min(E_3) = \Omega'^2 E = \Omega^2/4\sigma_1\sigma_2$.

As another example, starting at the other end of the $m' = 0$ line in figure 6

$$(E_1(0)\,\sigma_2 = E_2(0)\,\sigma_1, E_3(0) = 0),$$

it is $\Omega'^2 = (1-\sqrt{m})^2/\sqrt{m}, \mu^2 = \mu_0^2/\sqrt{m}$ and

$$\left.\begin{aligned} E_i(t) &= E_i(0)\,(1-\sqrt{m}\,\mathrm{sn}^2\mu t) \quad (i=1,2), \\ E_3(t) &= [(E_1(0)+E_2(0)]\sqrt{m}\,\mathrm{sn}^2\mu t. \end{aligned}\right\} \tag{5.27}$$

Thus for component 3 to gain at least half of the total energy, it is necessary that $|\Omega'| < 1/\sqrt{2}$. For $|\Omega'| = 1/\sqrt{2}$ it is $\mathrm{m} = \frac{1}{4}$ and the maximum of E_3 is reached in a time $t = K(\frac{1}{4})/\sqrt{2}\,\mu_0 = 1{\cdot}1920/\mu_0$.

If the total energy is large enough so that $\Omega' \sim 0$, then (5.26) and (5.27) tend to the solution of the resonant case, (5.23) with $m = 1$.

6. Discussion

The theory of nonlinear wave–wave interactions is revised here through the study of two different geophysical systems: barotropic Rossby waves (BRW) and internal gravity waves in a vertical plane (IGW). The main results are summarized below along with a discussion of their generality, i.e. a consideration of the necessary conditions for finding results similar to the ones found here, in other geophysical systems.

First of all, the nonlinear evolution equation in 'components' space is found without having to make the usual (and often cumbersome) multiple time-scale analysis or any other perturbation expansion. The simplicity of the method used in this paper represents an advantage over the perturbation expansion, especially for systems with many fields or represented by differential equations with non-constant coefficients (see for instance Ripa 1980*a*). Moreover, the evolution equation developed here (which has quadratic nonlinearity) is an *exact* representation of the original model equations, i.e. there are no extra terms missing, and its use is not constrained to motion of small amplitude.

The procedure is based on the use of the eigenfunctions of the linearized problem as a basis to expand the dynamical fields; thus, changing the description of the physical system from the dynamical fields to the expansion amplitudes, $Z_a(t)$. No assumption is made about the magnitude of fields (because the linearized equations are only used to provide the expansion basis), and the method is greatly simplified by the *orthogonality* of the expansion functions. A crucial property, *even for the linear problem*, is the *completeness* of the expansion basis, in order to make the representation in terms of the $Z_a(t)$ exact. Both properties are related to the *hermiticity* of the model equations; with a metric defined by the energy integral.

The same procedure can be applied to other geophysical systems as long as they have similar hermitian properties (Ripa 1980*a*); which is usually found to be true in the Eulerian description. Apparently, it is not possible to write the Lagrangian equations in an explicitly hermitian form, which is unfortunate because this seems to be the natural description for nonlinear problems with a free boundary, including the important case of surface waves.

As second result, the conservation laws of the systems are derived and related to the symmetries of the Lagrangian model equations. Both systems have several conservation laws, which can be written in terms of three independent ones. (1) A property of each fluid parcel is conserved (potential vorticity for BRW and total density for IGW). This law is related to invariance under a general change of the label of the fluid elements (which is taken to be equal to their equilibrium position), and serves

to define the inhomogeneous co-ordinate (y). (2) Total energy (E) is conserved; this is related to invariance under translations in time. The energy integral, which is positive definite, serves to define the metric in the space spanned by the linear eigenfunctions. (3) Total pseudomomentum P along x is conserved; this is related to invariance under translations along that co-ordinate, both in position and label spaces.

E and P are quadratic and additive functions of the amplitudes Z_a, and their conservation has, in turn, some interesting consequences, namely: (a) a generalization of Fjortoft's theorem, in x-slowness space instead of $|\mathbf{k}|$ space; (b) any combination of linear waves with the same phase speed along x (in particular, one single wave) turns out to be an exact solution of the full nonlinear problem; (c) E and P are still conserved after simplifying the system by means of some (arbitrary) truncation of the expansion basis (which is precisely what some numerical models do).

For other geophysical systems, conservation of E and of the total pseudo-momentum P_i along each homogeneous co-ordinate x_i are also expected.† E and the P_i should be, *to the lowest order*, quadratic and additive in the Z_a; with the pseudo-momentum of each component equal to its slowness vector times its energy. However, E and the P_i need not be exactly quadratic in the Z_a, and the consequences (a)–(c) of the last paragraph may not be true for finite-amplitude motion.

Take for instance the problem of IGW extended to 3 dimensions. The expansion set for this problem is divided into two classes: 3DIGW (which is the part of the motion with vanishing vertical vorticity) and sheared steady flow with vanishing vertical velocity. Both horizontal co-ordinates are homogeneous, symmetry related to the conservation of both horizontal pseudo-momenta. These are equal to

$$\langle \hat{\mathbf{z}} \times \boldsymbol{\zeta} \eta \rangle + O(Z_a^3),$$

where $\boldsymbol{\zeta}$ is the part of the horizontal vorticity due to the 3DIGW and η is the isopycnal elevation. Finally, this system is also invariant under rotations around any vertical axis, which results in another conservation law, namely that of the total pseudo-angular-momentum, equal to $\langle \mathbf{x} . \boldsymbol{\zeta} \eta \rangle + O(Z_a^3)$.

As a third result, the properties of all resonant triads are shown, and used to estimate the order of magnitude of the coupling coefficients of all off-resonant triads. The scale of the latter is given, for each interacting trio, by the total wavenumber of the component with intermediate x-slowness.

This information is used in the stability analysis of a single wave in order to estimate the value of the maximum growth rate, resulting in

$$\mu \sim |\mathbf{k}| \,\mathrm{rms}\, |\mathbf{v}|, \tag{6.1}$$

(where $\mathbf{k}$ is the wavenumber of the unstable wave) for IGW and short BRW. The value of μ for BRW much longer than the deformation radius or low-frequency IGW or BRW is smaller than (6.1). The actual values of the maximum growth rates are calculated for BRW (much shorter or longer than the deformation radius) and IGW, in both the strong- and weak-interactions limits, and for the extreme orientations of the wave. The calculated growth rates are essentially given by (6.1).

Finally, the complete solution of the three-wave problem is presented. In addition to the well known McGoldrick (1965) solutions with maximum energy exchange

† Note that the conservation of the pseudo-momentum along some coordinate is broken by the presence of boundaries not parallel to that direction.

(valid only in the strictly resonant case), new solutions are found with constant energies and shifted frequencies. In fact, these two sets of solutions are only extreme cases, since most solutions have properties 'in between' those of the former two, The nonlinear inverse time scale (for energy exchange) or the scale of the frequency shift is essentially given by (6.1) with **k** the wavenumber of the component with intermediate x-slowness.

This general solution is valid for both resonant and off-resonant cases, and is then used to show that the detuning inhibits energy exchange if it is larger than the nonlinear inverse time scale. Furthermore, this solution is also valid (in a range of the parameters different than the one discussed here) for other geophysical problems like the 'explosive' instability of a three-layer flow (Craik & Adam 1979).

A large portion of this work was done at the Geophysical Fluid Dynamics Laboratory in Princeton, where I benefited from the encouragement and advice of I. Orlanski. Many colleagues were very helpful in reading the original manuscript. I am particularly grateful to Greg Holloway and Ned Cockelet in this respect. I am also grateful to Fernando Lund for many discussions, to Ingrid Quetermous for typing the original manuscript and to D. Doyle for drafting the figures.

Appendix. Solution of the three-wave problem

Writing

$$A_i(t) = (\tfrac{2}{3})\,\mu_0^2[\cos(\alpha' - i2\pi/3) + C(t)], \tag{A 1}$$

for $i = 1, 2, 3$; the parameters on the right-hand side can be calculated as

$$\mu_0 = \operatorname{sgn}(\sigma_3)\,[A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_2A_3 - A_3A_1], \tag{A 2a}$$

$$\left.\begin{aligned} \tan\alpha' &= \sqrt{3}\,(A_1 - A_2)/(2A_3 - A_1 - A_2) \\ &\equiv m'\sqrt{(3)}/(2 - |m'|), \end{aligned}\right\} \tag{A 2b}$$

$$C(t) = (A_1 + A_2 + A_3)/2\mu_0^2. \tag{A 2c}$$

Both μ_0 and α' are constant, in virtue of (5.14)'; μ_0^2 may have values between $(\frac{3}{4})^{\frac{1}{3}}\,\sigma_1\sigma_2 E$ and $\max(-\sigma_{1,2}\sigma_3 E)$, whereas the possible values of α' are between $-\pi/3$ ($m' = -1$) and $\pi/3$ ($m' = 1$). The parameter m', used in the main text, is practically equal to $3\alpha'/\pi$. Finally, $C(t)$ may take any value between $-\cos\alpha'$ and $\cos(|\alpha'| + \pi/3)$.

Equation (5.12a) can be transformed, using (5.15), to

$$(\dot{C})^2/\mu_0^2 + (S' - 2C\Omega')^2 = (2/3)\,(\cos 3\alpha' + 4C^3 - 3C) \tag{A 3}$$

where $S' = 3S/\mu_0^3$ and $\Omega' = \Omega/2\mu_0$.

The solution of (A 3) is of the form (see Abramowitz & Stegun 1966, chapter 16)

$$C(t) = C_0 + C_1\,\mathrm{sn}\mu t.$$

The problem is then reduced to finding C_0, C_1, μ and the parameter m of the Jacobian sine from μ_0, α' and S'. This can be done using $(\mathrm{sn}'u)^2 = (1 - \mathrm{sn}^2u)\,(1 - m\,\mathrm{sn}^2u)$ and

equating the coefficients of the different powers of $\mathrm{sn}^2\mu t$ in both sides of (A 3). The solution is then given by (5.19) with

$$\cos 3\alpha = [3S'^2 - 6S'\Omega'^3 + 3\Omega'^2 - \Omega'^6 - 2\cos 3\alpha']/2|1 + \Omega'^4 - 2\Omega' S'|^{\frac{1}{2}},$$

$$0 \leqslant \alpha \leqslant \pi/3, \tag{A 4a}$$

$$m = 2\tan\alpha/(\sqrt{3} + \tan\alpha) \tag{A 4b}$$

$$\mu = \mu_0|1 + \Omega'^4 - 2\Omega' S'|^{\frac{1}{2}}(1 - m + m^2)^{-\frac{1}{4}} \tag{A 4c}$$

$$\lambda_i = \{1 + m - [2\cos(\alpha' - i2\pi/3) + \Omega'^2]\mu_0^2/\mu^2\}/3. \tag{A 4d}$$

The parameter m is practically equal to $3\alpha/\pi$. For a resonant triad, $\Omega' = 0, \mu/\mu_0$ may have any value between 1 and $(4/3)^{\frac{1}{2}}$ ($\simeq 1{\cdot}07$).

REFERENCES

Abramowitz, M. & Stegun, I. A. 1966 *Handbook of Mathematical Functions*. Dover.

Andrews, D. G. & McIntyre, M. E. 1978 On wave-action and its relatives. *J. Fluid Mech.* **89**, 647–664.

Bretherton, F. P. 1964 Resonant interactions between waves. The case of discrete oscillations. *J. Fluid Mech.* **20**, 457–479.

Craik, A. D. D. & Adam, J. A. 1979 'Explosive' resonant wave interactions in a three-layer fluid flow. *J. Fluid Mech.* **92**, 15–33.

Gelfand, I. M. & Fomin, S. V. 1963 *Calculus of Variations*. Prentice-Hall.

Gill, A. E. 1974 The stability of planetary waves on infinite beta-plane. *Geophys. Fluid Dyn.* **6**, 29–47.

Hasselmann, K. 1967 Nonlinear interactions treated by the methods of theoretical physics (with application to the generation of waves by wind). *Proc. Roy. Soc.* A **299**, 77–100.

Holloway, G. & Hendershott, M. C. 1977 Stochastic closure for nonlinear Rossby waves. *J. Fluid Mech.* **82**, 747–765.

Holloway, G. 1980 Oceanic internal waves are not weak waves. *J. Phys. Ocean.* **10**, 906–914.

Longuet-Higgins, M. S. & Gill, A. E. 1967 Resonant interactions between planetary waves. *Proc. Roy. Soc.* A **299**, 120–140.

Milder, M. 1976 A conservation law for internal gravity waves. *J. Fluid Mech.* **78**, 209–216.

McComas, C. H. & Bretherton, F. P. 1977 Resonant interaction of oceanic internal waves. *J. Geophys. Res.* **82**, 1397–1412.

McGoldrick, L. F. 1965 Resonant interactions among capillary–gravity waves. *J. Fluid Mech.* **21**, 305–331.

Orlanski, I. 1971 Energy spectrum of small-scale internal gravity waves. *J. Geophys. Res.* **76**, 5829–5835.

Orlanski, I. 1973 Trapeze instability as a source of internal gravity waves. Part 1. *J. Atmos. Sci.* **30**, 1007–1016.

Orlanski, I. & Cerasoli, C. P. 1980 Energy transfer among internal gravity modes: weak and strong interactions. *J. Geophys. Res.* (to appear).

Phillips, O. M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. *J. Fluid Mech.* **9**, 193–217.

Ripa, P. 1980*a* Nonlinear wave–wave interactions in the equatorial beta-plane. (In preparation.)

Ripa, P. 1980*b* Conservation laws for ocean models in the beta-plane and the sphere. (In preparation.)